On a parabolic–elliptic chemotactic system with non-constant chemotactic sensitivity

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A B S T R A C T

We study a parabolic–elliptic chemotactic system describing the evolution of a population's density "u" and a chemoattractant's concentration "v". The system considers a non-constant chemotactic sensitivity given by "$\chi(N-u)$", for $N \geq 0$, and a source term of logistic type "$\lambda u(1-u)$". The existence of global bounded classical solutions is proved for any $\chi > 0$, $N \geq 0$ and $\lambda \geq 0$. By using a comparison argument we analyze the stability of the constant steady state $u = 1$, $v = 1$, for a range of parameters.

– For $N > 1$ and $N\lambda > 2\chi$, any positive and bounded solution converges to the steady state.
– For $N \leq 1$ the steady state is locally asymptotically stable and for $\chi N < \lambda$, the steady state is globally asymptotically stable.

1. Introduction

Chemotaxis is the biological phenomenon whereby living organisms respond to a chemical substance by motion and rearrangement. One of the first mathematical models of chemotaxis was introduced by Keller and Segel [1] after Patlak [2]. The Patlak/Keller–Segel model considers a system of two parabolic equations while other authors have considered parabolic-ode or parabolic–elliptic systems of equations (see the review article of Horstmann [3] and the references therein for details). Keller and Segel [1] proposed a general model of partial differential equations for a population's density "u" and a chemoattractant's concentration "v"

$$\begin{cases} u_t = \nabla \cdot (\nabla u - u\chi(v)\nabla v) + g(u), \\ \epsilon v_t - \Delta v = f(u, v). \end{cases}$$

Hillen and Painter [4] consider an extension to the previous models introducing the effects of the finite size of individual cells and the employment of cell density sensing mechanisms "volume filling". The model has been formally derived by Painter and Hillen [5] for a nonlinear diffusion and a cross-diffusion term of the form $u\chi(u, v)\nabla v$ instead of the term $u\chi(v)\nabla v$ used in classical models; see also [6,16,17] and the reference therein for more details. The authors introduced a probabilistic approach to arrive at the following model

$$\begin{cases} u_t = \nabla \cdot \left[q(u) - uq'(u)\nabla u - uq(u)\chi(v)\nabla v\right] + g(u, v), \\ \epsilon v_t - \Delta v = f(u, v), \end{cases}$$

where “q” represents the probability that a cell finds space at its neighboring location. In [4] the authors consider the case

$$q(u) = D(N - u), \tag{1.1}$$

which gives a constant diffusion coefficient DN. Notice that up to the threshold value “N” the chemotaxis term is negative and the individuals move to a lower concentration of chemoattractant. The change of sign in q characterized the system which may evolve from positive to negative taxis or vice versa.

The growth term “g” in the first equation is defined by a logistic function and after normalization, g has the following expression

$$g(u) = \lambda u(1 - u). \tag{1.2}$$

Growth effects in chemotaxis systems has been considered to study the large time behavior. In absence of growth terms with constant chemosensitivity, the solution of the parabolic–elliptic system blows up at finite time in dimension 2 for a range of initial masses (see for instance Horstman [3] and Velázquez [7]). Growth terms may prevent blow up in chemotaxis systems, as shown by the numerous examples existing in the literature. For instance, in Osaki, Tsujikawa, Yagi and Mimura [8], the logistic growth in a two dimensional parabolic–parabolic chemotaxis system drives the solution to an exponential attractor in a suitable space. In Winkler [9], growth terms $f \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ satisfying $f(s) \leq a - \mu s^2$ for $\mu > \mu_0(a)$ shows global existence of solutions with no dimensional restrictions, i.e. $n \geq 1$. Similar result for the parabolic–elliptic problem can be found in Mimura and Tsujikawa [10] (see also [11] and [15]).

Notice that the threshold value in the chemotaxis and logistic terms N and 1 are not necessarily equal. The sign of the difference of these values gives a different analysis of the stability of the constant steady states (see Section 3 in present paper for details).

Different authors consider a fast diffusion process for the chemoattractant substance and simplify the parabolic equation describing the evolution of v by an elliptic equation taking $\epsilon = 0$ (see for instance Velázquez [7] or Wang, Winkler and Wrzosek [12]). The equation is simplified by the following one

$$-\Delta v = f(u, v).$$

In numerous biologically relevant processes, the chemical substance is produced by the individuals of the population and the function f satisfies

$$\frac{\partial f}{\partial u} > 0.$$

As in the classical Keller–Segel system we consider a degradation of v and simplify the term f by the linear expression $f(u, v) = f_0 u - f_1 v$ and without loss of generality we assume $f_0 = f_1 = 1$. Then the distribution of chemoattractant is governed by the linear elliptic equation of the form

$$-\Delta v + v = u.$$

We consider a “volume filling” model with fast diffusion process for the chemical substance with logistic growth term. The problem is given by a parabolic–elliptic system defined over a bounded domain Ω with regular boundary $\partial\Omega$:

$$\begin{cases} u_t - \Delta u = -\chi \nabla \cdot (u(N - u)\nabla v) + \lambda u(1 - u), & x \in \Omega, t > 0, \\ -\Delta v + v = u, & x \in \Omega, t > 0, \end{cases} \tag{1.3}$$

with the Neumann boundary conditions

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad x \in \partial\Omega,\ t > 0 \tag{1.4}$$

and the initial data

$$u(0, x) = u_0(x), \quad x \in \Omega. \tag{1.5}$$

We also consider that the initial data satisfies

$$u_0 \in C^{2,\alpha}(\overline{\Omega}) \quad \text{and} \quad \frac{\partial u_0}{\partial n} = 0.$$

The main result of the paper is the asymptotic stability of the constant steady state $u = v = 1$, for a range of parameters and initial data u_0. The result is enclosed in the following theorem.

Theorem 1.1. *Let $\beta > 0$ such that $\beta \leq u_0$, for $x \in \Omega$. Then*

1. *if $N > 1$ and $\lambda N > 2\chi$,*

2. *if* $N \le 1$ *and either:*

$$\lambda > \chi \quad \text{or} \quad \beta \ge N \quad \text{or} \quad \beta - \max\{\max_{x\in\Omega}\{u_0\}, 1\}\chi + \lambda > 0$$

then, the solution (u, v) *to* (1.3) *satisfies*

$$\|u-1\|_{L^\infty(\Omega)} + \|v-1\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty.$$

In order to proof the theorem and analyze the stability of the problem, we introduce two auxiliary functions $\overline{u}$, $\underline{u}$ as the solutions of a system of ODE's. Since standard comparison arguments cannot be applied due to the sign variability of the chemotactic sensitivity, we introduce a comparison argument to obtain $\underline{u} \le u \le \overline{u}$ in Section 3. In Section 3.5 we analyze the system of ODE's to obtain the asymptotic behavior of the barrier functions. In Section 4 the existence and uniqueness of solutions is presented using the results of Section 3.5 as a priori estimates. The paper ends with a corollary of Theorem 1.1 concerning the steady states of the system.

Remark 1.2. If the logistic term in (1.3) is replaced by a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $g(0) \ge 0$ and there exists $\lambda \ge 0$ such that

$$(g(x) - g(y))\text{sign}(x-y) \le \lambda|x-y|, \quad \text{for any } x, y \in \mathbb{R}_+, \tag{1.6}$$

for some $\lambda \ge 0$, then, Theorem 3.1 is valid if the system of ODEs (2.1) is replaced by

$$\begin{cases} \overline{u}_t = \chi\overline{u}(N-\overline{u})(\overline{u} - \phi_1(\underline{u}, \overline{u})) + g(\overline{u}), & t > 0, \\ \underline{u}_t = \chi\underline{u}(N-\underline{u})(\underline{u} - \phi_2(\underline{u}, \overline{u})) + g(\underline{u}), & t > 0, \\ 0 < \underline{u}_0 \le u_0 \le \overline{u}_0 < \infty, \end{cases}$$

where ϕ_1 and ϕ_2 are defined in (2.3) and (2.4) respectively. To obtain a similar result that in Theorem 1.1, the solution of the previous ODE system has to satisfy

$$\overline{u}, \underline{u} \to 1, \quad \text{as } t \to \infty.$$

2. Analysis of the associated ODE system

In this section we consider the system of ODE's associated to the nonlinear system of PDE's

$$\begin{cases} \overline{u}_t = \chi\overline{u}(N-\overline{u})(\overline{u} - \phi_1(\underline{u}, \overline{u})) + \lambda\overline{u}(1-\overline{u}), & t > 0, \\ \underline{u}_t = \chi\underline{u}(N-\underline{u})(\underline{u} - \phi_2(\underline{u}, \overline{u})) + \lambda\underline{u}(1-\underline{u}), & t > 0, \end{cases} \tag{2.1}$$

with initial conditions

$$\overline{u}(0) = \overline{u}_0, \qquad \underline{u}(0) = \underline{u}_0, \tag{2.2}$$

where $\phi_1(\cdot,\cdot)$ and $\phi_2(\cdot,\cdot)$ are defined by

$$\phi_1(\underline{u}, \overline{u}) = \begin{cases} \underline{u} & \text{if } \overline{u} < N, \\ \overline{u} & \text{if } \overline{u} \ge N, \end{cases} \tag{2.3}$$

$$\phi_2(\underline{u}, \overline{u}) = \begin{cases} \overline{u} & \text{if } \underline{u} < N, \\ \underline{u} & \text{if } \underline{u} \ge N. \end{cases} \tag{2.4}$$

To begin with, let us make sure that the initial ordering $0 < \underline{u}_0 < \overline{u}_0$ is inherited by the solution. Moreover we shall prove that $(\underline{u}, \overline{u})$ is actually global in time and bounded, results which we present in the following lemma.

Lemma 2.1. *The solution to the system* (2.1)–(2.2) *exists in* $(0, \infty)$. *Moreover, under the assumption*

$$0 < \underline{u}_0 < \overline{u}_0 < \infty \tag{2.5}$$

the solution satisfies

$$0 < \underline{u} < \overline{u} \le \max\{\overline{u}_0, N, 1\} \quad \textit{for any } t < \infty. \tag{2.6}$$

Proof. It is easy to observe that the functions

$$\chi\overline{u}(N-\overline{u})(\overline{u} - \phi_1(\underline{u}, \overline{u})) + \lambda\overline{u}(1-\overline{u})$$

and

$$\chi\underline{u}(N-\underline{u})(\underline{u} - \phi_2(\underline{u}, \overline{u})) + \lambda\underline{u}(1-\underline{u})$$

are continuous and locally Lipschitz in $\underline{u}$ and $\overline{u}$. In fact, (2.1)–(2.2) is locally well posed and there exists an unique local solution for $t \in (0, T_{\max})$ such that, if $T_{\max} < \infty$, we have $|\overline{u}(T_{\max})| + |\underline{u}(T_{\max})| = \infty$.

Since $\max\{\overline{u}_0, N, 1\}$ is a super-solution to the first equation and $\underline{u} = 0$ is a sub-solution to the second equation in (2.1), we have, by uniqueness of solutions, that

$$0 < \underline{u}, \qquad \overline{u} \le \max\{\overline{u}_0, N, 1\}.$$

To prove $\underline{u} < \overline{u}$, we argue by contradiction. Hence, if $\underline{u} < \overline{u}$ is false, then, there exist some positive $t_0 < T_{\max}$ such that

$$\underline{u}(t_0) = \overline{u}(t_0), \qquad \underline{u}(t) < \overline{u}(t) \quad \text{for } t < t_0. \tag{2.7}$$

The solution to (2.1)–(2.2), with initial data $\underline{u}(t_0) = \overline{u}(t_0)$, satisfies $\underline{u} = \overline{u}$ for any $t > t_0$. We extend such solution to $(t_0 - \epsilon, t_0)$ to have $\underline{u} = \overline{u}$ for $t \in (t_0 - \epsilon, t_0 + \epsilon)$ which contradicts (2.7) and proves that

$$\underline{u}(t) < \overline{u}(t) \quad \text{for } t \in (0, T_{\max}). \tag{2.8}$$

(2.7) and (2.8) prove $T_{\max} = \infty$ and concludes the proof. □

In order to analyze the system (2.1)–(2.2) we consider two different cases, $N \le 1$ and $N > 1$.

Lemma 2.2. *Let us assume that $N \le 1$ and $\underline{u}_0 < 1$, then:*

1. *Under the assumption $\overline{u}_0 \ge 1$ we find that the solution $\overline{u}$ to the first equation in (2.1) is given by*

$$\overline{u}(t) = \frac{\overline{u}_0 e^{\lambda t}}{1 + \overline{u}_0(e^{\lambda t} - 1)}. \tag{2.9}$$

2. *If $\overline{u}_0 \ge 1$ and $\lambda > \chi$, then*

$$\underline{u} \to 1 \quad \text{as } t \to \infty. \tag{2.10}$$

3. *Let $\overline{u}_0 \ge 1 \ge \underline{u}_0 > 0$ then, if*

$$\underline{u}_0 \ge N, \tag{2.11}$$

or

$$(\underline{u}_0 - \max\{\overline{u}_0, 1\})\chi + \lambda > 0, \tag{2.12}$$

we have

$$\underline{u} \to 1 \quad \text{as } t \to \infty. \tag{2.13}$$

Proof. 1. Since $\overline{u}_0 \ge 1 \ge N$ we have that $\phi_1(\overline{u}) = \overline{u}$ and then, $\overline{u}$ satisfies

$$\overline{u}_t = \lambda \overline{u}(1 - \overline{u}),$$

which solution is given by (2.9).

2. Notice that $\underline{u}$ satisfies

$$\underline{u}_t = \chi \underline{u}(N - \underline{u})(\underline{u} - \overline{u}) + \lambda \underline{u}(1 - \underline{u}),$$

which implies

$$\underline{u}_t \ge \underline{u}(1 - \underline{u})(\chi(\underline{u} - \overline{u}) + \lambda) \ge \underline{u}(1 - \underline{u})(\lambda - \chi \overline{u}). \tag{2.14}$$

Since $\overline{u} = \frac{\overline{u}_0 e^{\lambda t}}{1 + \overline{u}_0(e^{\lambda t} - 1)}$, there exists $t_0 > 0$, such that $1 - \overline{u} \le \frac{\lambda - \chi}{2\chi}$ and this is equivalent to

$$\underline{u}(1 - \underline{u})(\lambda - \chi \overline{u}) \ge \underline{u}(1 - \underline{u})\frac{\lambda - \chi}{2}, \quad \text{for } t > t_0. \tag{2.15}$$

In view of (2.14) and (2.15), we have

$$\underline{u}_t \ge \underline{u}(1 - \underline{u})\frac{\lambda - \chi}{2}, \quad \text{for } t > t_0,$$

and therefore

$$\lim_{t \to \infty} \underline{u} \ge 1.$$

Eq. (2.6) and Lemma 2.1 end the proof in this case.

3. We consider two different cases: $\underline{u}_0 \geq N$ and $\underline{u}_0 < N$
   - If $\underline{u}_0 \geq N$, the solution satisfies

     $$\underline{u}_t = \lambda \underline{u}(1 - \underline{u})$$

     and the solution is given by

     $$\underline{u} = \frac{\underline{u}_0 e^{\lambda t}}{1 + \underline{u}_0(e^{\lambda t} - 1)},$$

     which satisfies (2.13).
   - If $\underline{u}_0 < N$, we have

     $$\underline{u}_t = \chi \underline{u}(N - \underline{u})(\underline{u} - \overline{u}) + \lambda \underline{u}(1 - \underline{u})$$

     and since $\overline{u} = \frac{\overline{u}_0 e^{\lambda t}}{1+\overline{u}_0(e^{\lambda t}-1)}$, which is a monotone decreasing function, thus, as far as $\underline{u} \leq N$,

     $$\underline{u}_t \geq \chi \underline{u}(N - \underline{u})(\underline{u} - \overline{u}_0) + \lambda \underline{u}(1 - \underline{u}),$$

     which implies

     $$\underline{u}_t \geq \underline{u}(1 - \underline{u})(\lambda + \chi(\underline{u} - \overline{u}_0)).$$

     Since $\underline{u}_0$ satisfies (2.12), we have $\underline{u}_t|_{t=0} > 0$ and therefore

     $$\lambda + \chi(\underline{u} - \overline{u}_0) \geq \lambda + \chi(\underline{u}_0 - \overline{u}_0) > 0,$$

     and then $\underline{u}$ satisfies

     $$\underline{u}_t \geq \epsilon \underline{u}(1 - \underline{u}),$$

     as far as $\underline{u} \leq N$, for $\epsilon := \lambda + \chi(\underline{u}_0 - \overline{u}_0) \leq \lambda$and

     $$\underline{u} = \frac{\underline{u}_0 e^{\epsilon t}}{1 + \underline{u}_0(e^{\epsilon t} - 1)},$$

     as far as $\underline{u} \leq N$. If $N < 1$, there exists $t_0 < \infty$ such that $\underline{u} = N$, $\underline{u}_t = \lambda \underline{u}(1 - \underline{u})$, for $t \geq t_0$ as we wanted to prove. □

Lemma 2.3. *We assume that $N > 1$.*

1. *There exists $t_0 \geq 0$ such that $\overline{u}(t_0) \leq N$.*
2. *Let $\overline{u}_0 \in [1, N]$ and $0 < \underline{u}_0 < \overline{u}_0 < \infty$. Then, under the assumption*

   $$\lambda N > 2\chi, \tag{2.16}$$

   we have

   $$\overline{u}, \underline{u} \to 1 \quad \textit{as } t \to \infty. \tag{2.17}$$

Proof. 1. Since the case $\overline{u}_0 \leq N$ is trivial, we just consider the case $\overline{u}_0 > N$, where $\overline{u}$ satisfies

$$\overline{u}_t = \lambda \overline{u}(1 - \overline{u})$$

as far as $\overline{u} \geq N$. Therefore we have

$$\overline{u} = \frac{\overline{u}_0 e^{\lambda t}}{1 + \overline{u}_0(e^{\lambda t} - 1)} \quad \text{for } t \leq \frac{1}{\lambda} \ln \frac{N(\overline{u}_0 - 1)}{\overline{u}_0(N - 1)}.$$

Then $\overline{u} = N$ for $t = \frac{1}{\lambda} \ln \frac{N(\overline{u}_0-1)}{\overline{u}_0(N-1)}$, and the proof in the first case finishes.

Notice that it is enough to consider the case $\overline{u}_0 \leq N$, which is study the next part of the proof.

2. Since $\overline{u}_0 > \underline{u}_0$ we have

$$\overline{u}_t = \chi \overline{u}(N - \overline{u})(\overline{u} - \underline{u}) + \lambda \overline{u}(1 - \overline{u}), \tag{2.18}$$

$$\underline{u}_t = \chi \underline{u}(N - \underline{u})(\underline{u} - \overline{u}) + \lambda \underline{u}(1 - \underline{u}). \tag{2.19}$$

Notice that by (2.5), $\overline{u} = N$ is a supersolution to (2.18), and therefore

$$\overline{u}_t \leq \chi N \overline{u}(\overline{u} - \underline{u}) + \lambda \overline{u}(1 - \overline{u}). \tag{2.20}$$

In the same way we have

$$\underline{u}_t \geq \chi N \underline{u}(\underline{u} - \overline{u}) + \lambda \underline{u}(1 - \underline{u}). \tag{2.21}$$

System (2.20), (2.21) is treated as in [11]. Since $0 < \underline{u} < \overline{u}$ it results

$$\frac{\overline{u}_t}{\overline{u}} \le \chi N(\overline{u} - \underline{u}) + \lambda(1 - \overline{u}),$$
$$\frac{\underline{u}_t}{\underline{u}} \ge \chi N(\underline{u} - \overline{u}) + \lambda(1 - \underline{u}).$$

We subtract both equations to obtain

$$\frac{d}{dt}\left[\ln \overline{u} - \ln \underline{u}\right] \le (2\chi N - \lambda)(\overline{u} - \underline{u}). \tag{2.22}$$

After integration over $(0, t)$, we get that

$$\ln \frac{\overline{u}}{\underline{u}} \le \ln \frac{\overline{u}_0}{\underline{u}_0},$$

i.e.,

$$\frac{\underline{u}_0}{\overline{u}_0}\overline{u} \le \underline{u}.$$

Since the initial data satisfies $\overline{u}_0 \ge 1$ we notice that

$$\overline{u} \ge 1. \tag{2.23}$$

Therefore, we have

$$\frac{\underline{u}_0}{\overline{u}_0} \le \underline{u}. \tag{2.24}$$

Thanks to (2.22), (2.24) and Mean Value Theorem, we obtain

$$\frac{d}{dt}\left[\ln \overline{u} - \ln \underline{u}\right] \le (2\chi N - \lambda)\frac{\underline{u}_0}{\overline{u}_0}(\ln \overline{u} - \ln \underline{u}) \tag{2.25}$$

and after integration we conclude

$$\left[\ln \overline{u} - \ln \underline{u}\right] \to 0,$$

and thanks to (2.23) the proof ends. □

3. Comparison argument

In this section we detail the computations of the comparison argument which establishes the connection between (2.1)–(2.2) and (1.3).

Theorem 3.1. *Let $u_0 \in L^\infty(\Omega)$ and $\beta > 0$ such that*

$$\beta \le \underline{u}_0 \le u_0 \le \overline{u}_0 \quad \text{in } \Omega.$$

Then, the solution (u, v) of (1.3) fulfills

$$\underline{u} \le u \le \overline{u}, \qquad \underline{u} \le v \le \overline{u} \quad (x, t) \in \Omega \times (0, \infty). \tag{3.1}$$

In order to prove the theorem we introduce the following notations:

$$\begin{aligned} \overline{U}(x,t) &:= u(x,t) - \overline{u}(t), \qquad \underline{U}(x,t) := u(x,t) - \underline{u}(t), \\ \overline{V}(x,t) &:= v(x,t) - \overline{u}(t), \qquad \underline{V}(x,t) := v(x,t) - \underline{u}(t), \end{aligned} \tag{3.2}$$

and the standard positive and negative part functions:

$$(s)_+ = \begin{cases} s & \text{if } s \ge 0, \\ 0 & \text{otherwise} \end{cases} \qquad (s)_- = (-s)_+.$$

Notice that (1.3) is equivalent to

$$\begin{cases} u_t - \Delta u = -\chi(N - 2u)\nabla u \cdot \nabla v + \chi u(N - u)(u - v) + g(u) \quad \text{in } \Omega_T, \\ -\Delta v + v = u \quad \text{in } \Omega_T, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, \quad \text{in } \partial\Omega, \\ u(0, x) = u_0(x) \quad \text{in } \Omega. \end{cases} \tag{3.3}$$

Then $\overline{U}$ satisfies the following PDE

$$\overline{U}_t - \Delta\overline{U} = -\chi(N-2u)\nabla\overline{U}\cdot\nabla v + \chi u(N-u)(u-v) - \chi\overline{u}(N-\overline{u})(\overline{u}-\phi_1(\overline{u},\underline{u})) + g(u) - g(\overline{u}). \tag{3.4}$$

We take $\overline{U}_+^{s-1}$ as test function in (3.4), i.e., multiply by $\overline{U}_+^{s-1}$ for

$$s := \max\left\{\frac{n}{2}+1, 7\right\}. \tag{3.5}$$

For technical reasons due to Sobolev embeddings we have to consider $s > \frac{n}{2}$, that will be detailed at the end of the proof. We integrate by parts over Ω to obtain, after some routinary computations:

$$\begin{aligned}\frac{1}{s}\frac{d}{dt}\int_\Omega \overline{U}_+^s + (s-1)\int_\Omega \overline{U}_+^{s-2}|\nabla\overline{U}_+|^2 &= -\chi\int_\Omega \overline{U}_+^{s-1}(N-2u)\nabla\overline{U}_+\cdot\nabla v + \int_\Omega \overline{U}_+^{s-1}(g(u)-g(\overline{u}))\\ &\quad + \chi\int_\Omega \overline{U}_+^{s-1}[u(N-u)(u-v) - \overline{u}(N-\overline{u})(\overline{u}-\phi_1(\overline{u},\underline{u}))].\end{aligned} \tag{3.6}$$

In order to prove the theorem we consider the following technical lemma:

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^n$ for $p \in \mathbb{N}$, $p \in (\max\{\frac{n}{2}, 1\}, \infty)$ and v the solution to*

$$\begin{cases} -\Delta v + v = u, & x \in \Omega,\\ \dfrac{\partial v}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{3.7}$$

for $u \in L^p(\Omega)$. Then, for any $q \le \infty$ the following inequalities holds:

$$\|\overline{V}_+\|_{L^q(\Omega)} \le C(\Omega, q, p)\|\overline{U}_+\|_{L^p(\Omega)} \tag{3.8}$$

and

$$\|\overline{V}_-\|_{L^q(\Omega)} \le C(\Omega, q, p)\|\overline{U}_-\|_{L^p(\Omega)}. \tag{3.9}$$

Proof. Thanks to (3.2) we may rewrite (3.7) as follows

$$\begin{cases} -\Delta\overline{V} + \overline{V} = \overline{U}, & x \in \Omega,\\ \dfrac{\partial\overline{V}}{\partial n} = 0, & x \in \partial\Omega. \end{cases}$$

We consider now V_1, the solution to the problem

$$\begin{cases} -\Delta V_1 + V_1 = \overline{U}_+, & x \in \Omega,\\ \dfrac{\partial V_1}{\partial n} = 0, & x \in \partial\Omega, \end{cases}$$

and apply maximum principle to have that $V_1 \ge 0$. Since $\overline{U}_+ \in L^p(\Omega)$ we obtain

$$\|V_1\|_{W^{2,p}(\Omega)} \le C_1(\Omega)\|\overline{U}_+\|_{L^p(\Omega)},$$

thanks to the embedding $W^{2,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ for $p > \frac{n}{2}$ we have that,

$$\|V_1\|_{L^q(\Omega)} \le C_2(\Omega)\|V_1\|_{W^{2,p}(\Omega)} \le C_3(\Omega)\|\overline{U}_+\|_{L^p(\Omega)} \quad \text{for any } q \le \infty. \tag{3.10}$$

Since $\overline{U} \le \overline{U}_+$, by maximum principle we get that $\overline{V} \le V_1$ and therefore

$$0 \le \overline{V}_+ \le (V_1)_+ = V_1,$$

which implies,

$$\|\overline{V}_+\|_{L^q(\Omega)} \le \|V_1\|_{L^q(\Omega)} \quad \text{for any } q < \infty. \tag{3.11}$$

Thanks to (3.10) and (3.11) we have (3.8). The same argument proves (3.9) and the proof ends. □

Lemma 3.3. *For any $\epsilon > 0$ arbitrary there exists a positive constant $k(\epsilon)$ such that, with the above notations, the below inequality holds*

$$\begin{aligned}&\int_\Omega [\chi u(N-u)(u-v) - \chi\overline{u}(N-\overline{u})(\overline{u}-\phi_1(\overline{u},\underline{u}))]\overline{U}_+^{s-1}\\ &\quad \le -\chi(1-\epsilon)\int_\Omega \overline{U}_+^{s+2} + k(\epsilon)\int_\Omega \overline{U}_+^s + k(\epsilon)\int_\Omega [\overline{V}_+^{s\frac{s+1}{s-1}} + \overline{V}_+^{s+2} + \underline{V}_-^{s+2} + \overline{V}_+^{s+1} + \overline{V}_+^s + \underline{V}_-^s].\end{aligned} \tag{3.12}$$

Proof. Since $u = \overline{U} + \overline{u}$, we have

$$\begin{aligned} u(N-u)(u-v) - \overline{u}(N-\overline{u})(\overline{u} - \phi_1(\overline{u}, \underline{u})) &= \overline{U}(N-\overline{U})(u-v) - \overline{U}\overline{u}(u-v) + \overline{u}(N-\overline{U})(u-v) \\ &\quad - \overline{u}^2(u-v) - \overline{u}(N-\overline{u})(\overline{u} - \phi_1(\overline{u}, \underline{u})) \end{aligned}$$

and after some computations, we obtain that

$$u(N-u)(u-v) - \overline{u}(N-\overline{u})(\overline{u} - \phi_1(\overline{u}, \underline{u})) = \overline{U}(N - \overline{U} - 2\overline{u})(u-v) + \overline{u}(N-\overline{u})[\overline{U} - v + \phi_1].$$

Taking into account that $u - v = \overline{U} - \overline{V}$, we deduce the following

$$\begin{aligned} &\overline{U}_+^{s-1}[\overline{U}(N - \overline{U} - 2\overline{u})(\overline{U} - \overline{V}) + \overline{u}(N-\overline{u})(\overline{U} - (v - \phi_1))] \\ &\quad = \overline{U}_+^s(N - \overline{U} - 2\overline{u})(\overline{U} - \overline{V}) + \overline{u}(N-\overline{u})(\overline{U}_+^s - (v-\phi_1)\overline{U}_+^{s-1}). \end{aligned}$$

In order to bound the terms in the above equation, we distinguish three different cases:
Case I. $\overline{u} \geq N$.
In this case, $N - \overline{u} \leq 0$ and $\phi_1 = \overline{u}$, then

$$\begin{aligned} &\overline{U}_+^s(N - \overline{U} - 2\overline{u})(\overline{U} - \overline{V}) + \overline{u}(N-\overline{u})(\overline{U}_+^s - (v-\phi_1)\overline{U}_+^{s-1}) \\ &\quad = \overline{U}_+^s(N - \overline{U} - 2\overline{u})(\overline{U} - \overline{V}) + \overline{u}(N-\overline{u})(\overline{U}_+^s - \overline{V}\,\overline{U}_+^{s-1}). \end{aligned} \tag{3.13}$$

Notice that since $\overline{U}_+\overline{U} = \overline{U}_+^2$ we have

$$\overline{U}_+^s(N - \overline{U} - 2\overline{u})(\overline{U} - \overline{V}) = \overline{U}_+^s(N - \overline{U}_+ - 2\overline{u})(\overline{U}_+ - \overline{V}).$$

We apply the positive part function to the term which contains $\overline{V}$ to obtain

$$-\overline{U}_+^s(N - \overline{U}_+ - 2\overline{u})\overline{V} \leq [-\overline{U}_+^s(N - \overline{U}_+ - 2\overline{u})\overline{V}]_+.$$

In this case $(N - \overline{u} \leq 0)$ we deduce that $N - \overline{U}_+ - 2\overline{u} \leq 0$ to get

$$[-\overline{U}_+^s(N - \overline{U}_+ - 2\overline{u})\overline{V}]_+ = -\overline{U}_+^s(N - \overline{U}_+ - 2\overline{u})\overline{V}_+ \tag{3.14}$$

which gives

$$\overline{U}_+^s(N - \overline{U}_+ - 2\overline{u})(\overline{U}_+ - \overline{V}) \leq \overline{U}_+^s(N - \overline{U}_+ - 2\overline{u})(\overline{U}_+ - \overline{V}_+).$$

In the second term of (3.13), since $\overline{u}(N - \overline{u}) \leq 0$ we have

$$\overline{u}(N-\overline{u})(\overline{U}_+^s - \overline{V}\overline{U}_+^{s-1}) \leq \overline{u}(N-\overline{u})(\overline{U}_+^s - \overline{V}_+\overline{U}_+^{s-1}).$$

Therefore

$$\begin{aligned} &\overline{U}_+^s(N - \overline{U} - 2\overline{u})(\overline{U} - \overline{V}) + \overline{u}(N-\overline{u})(\overline{U}_+^s - (v-\phi_1)\overline{U}_+^{s-1}) \\ &\quad \leq \overline{U}_+^s(N - \overline{U}_+ - 2\overline{u})(\overline{U}_+ - \overline{V}_+) + \overline{u}(N-\overline{u})(\overline{U}_+^s - \overline{V}_+\overline{U}_+^{s-1}) \\ &\quad = -\overline{U}_+^{s+2} + \overline{U}_+^{s+1}(N - 2\overline{u} + \overline{V}_+) - \overline{U}_+^s(N - 2\overline{u})\overline{V}_+ + \overline{u}(N-\overline{u})\overline{U}_+^s - \overline{u}(N-\overline{u})\overline{V}_+\overline{U}_+^{s-1}. \end{aligned}$$

Since

$$\overline{U}_+^s\overline{V}_+ \leq \overline{U}_+^{s+1} + k_1(s)\overline{V}_+^{s+1},$$

then

$$\overline{U}_+^{s+1}(N - 2\overline{u}) - \overline{U}_+^s(N - 2\overline{u})\overline{V}_+ \leq k(s, \|\overline{u}\|_{L^\infty})\overline{V}_+^{s+1}.$$

Thanks to the previous inequality and

$$\overline{U}_+^{s+1}\overline{V}_+ \leq \epsilon\overline{U}_+^{s+2} + k(\epsilon)\overline{V}_+^{s+2}, \qquad \overline{U}_+^{s-1}\overline{V}_+ \leq \frac{s-1}{s}\overline{U}_+^s + \frac{1}{s}\overline{V}_+^s \tag{3.15}$$

we have

$$\begin{aligned} &-\overline{U}_+^{s+2} + \overline{U}_+^{s+1}(N - 2\overline{u} + \overline{V}_+) - \overline{U}_+^s(N - 2\overline{u})\overline{V}_+ + \overline{u}(N-\overline{u})\overline{U}_+^s - \overline{u}(N-\overline{u})\overline{V}_+\overline{U}_+^{s-1} \\ &\quad \leq -(1-\epsilon)\overline{U}_+^{s+2} + k(\epsilon)\overline{V}_+^{s+2}k_2(s)\left(\overline{V}_+^{s+1} + \overline{V}_+^s + \overline{U}_+^s\right). \end{aligned}$$

Then, for $N-\overline{u}\le 0$, we obtain that

$$\int_\Omega [\chi u(N-u)(u-v)-\chi\overline{u}(N-\overline{u})(\overline{u}-\phi_1(\overline{u},\underline{u}))]\overline{U}_+^{s-1}$$
$$\le -\chi(1-\epsilon)\int_\Omega \overline{U}_+^{s+2}+k(\epsilon)\int_\Omega \overline{V}_+^{s+2}+k(s)\int_\Omega \overline{V}_+^{s+1}+\overline{V}_+^{s}+\overline{U}_+^{s}. \tag{3.16}$$

Case II. $N-\overline{u}>0$ and $\overline{u}+u\le N$.

In this case we have

$$\overline{u}(N-\overline{u})>0,\qquad N-\overline{U}-2\overline{u}=N-u-\overline{u}\le 0,\qquad \phi_1=\underline{u}$$

and

$$\overline{U}_+^s(N-\overline{U}-2\overline{u})(\overline{U}-\overline{V})+\overline{u}(N-\overline{u})(\overline{U}_+^s-(v-\phi_1)\overline{U}_+^{s-1})$$
$$=\overline{U}_+^s(N-\overline{U}-2\overline{u})(\overline{U}-\overline{V})+\overline{u}(N-\overline{u})(\overline{U}_+^s-\underline{V}\overline{U}_+^{s-1}). \tag{3.17}$$

As in Case I, we know that $\overline{U}_+\overline{U}=\overline{U}_+^2$ and therefore

$$\overline{U}_+^s(N-\overline{U}-2\overline{u})(\overline{U}-\overline{V})+\overline{u}(N-\overline{u})(\overline{U}_+^s-\underline{V}\overline{U}_+^{s-1})$$
$$=\overline{U}_+^s(N-\overline{U}_+-2\overline{u})(\overline{U}_+-\overline{V})+\overline{u}(N-\overline{u})(\overline{U}_+^s-\underline{V}\overline{U}_+^{s-1}).$$

The term $-\overline{U}_+^s(N-\overline{U}_+-2\overline{u})\overline{V}$ is treated in the following way

$$\begin{aligned}\overline{U}_+^s(N-\overline{U}_+-2\overline{u})(\overline{U}_+-\overline{V}) &= \overline{U}_+^s(N-\overline{U}_+-2\overline{u})(\overline{U}_+-\underline{V}+\overline{u}-\underline{u})\\ &\le \overline{U}_+^s(N-\overline{U}_+-2\overline{u})(\overline{U}_++\underline{V}_-+\overline{u}-\underline{u}).\end{aligned}$$

Since $N-\overline{u}\ge 0$ we have that the last term in (3.17) is bound by

$$\overline{u}(N-\overline{u})(\overline{U}_+^s-\underline{V}\overline{U}_+^{s-1})\le \overline{u}(N-\overline{u})(\overline{U}_+^s+\underline{V}_-\overline{U}_+^{s-1})$$

and therefore

$$\overline{U}_+^s(N-\overline{U}-2\overline{u})(\overline{U}-\overline{V})+\overline{u}(N-\overline{u})(\overline{U}_+^s-\underline{V}\overline{U}_+^{s-1})$$
$$\le \overline{U}_+^s(N-\overline{U}_+-2\overline{u})(\overline{U}_++\underline{V}_-+\overline{u}-\underline{u})+\overline{u}(N-\overline{u})(\overline{U}_+^s+\underline{V}_-\overline{U}_+^{s-1}).$$

Notice that since $N-\overline{U}_+-2\overline{u}\le N$ and $\overline{u}-\underline{u}\le N$ we have that

$$\overline{U}_+^s(N-\overline{U}_+-2\overline{u})(\underline{V}_-+\overline{u}-\underline{u})\le k_1(\overline{U}_+^s+\overline{U}_+^s\underline{V}_-).$$

Thanks to the positivity of $\overline{u}$ and $u+\overline{u}\le N$ we have that $\overline{U}_+\le N$ which implies

$$k_1(\overline{U}_+^s+\overline{U}_+^s\underline{V}_-)\le k_1(\overline{U}_+^s+N\overline{U}_+^{s-1}\underline{V}_-)$$

and

$$\overline{U}_+^s(N-\overline{U}_+-2\overline{u})\overline{U}_+\le -\overline{U}_+^{s+2}+N^2\overline{U}_+^s.$$

To conclude this case, we proceed as in case I. By (3.15), it results

$$\int_\Omega [\chi u(N-u)(u-v)-\chi\overline{u}(N-\overline{u})(\overline{u}-\phi_1(\overline{u},\underline{u}))]\overline{U}_+^{s-1}\le -\chi\int_\Omega \overline{U}_+^{s+2}+k\int_\Omega \overline{U}_+^s+\overline{V}_-^s, \tag{3.18}$$

for some positive constant k.

Case III. $N-\overline{u}>0$ and $\overline{u}+u\ge N$.

In this case, we have $\overline{u}(N-\overline{u})>0$ and $\phi_1=\underline{u}$, then:

$$\overline{U}_+^s(N-\overline{U}-2\overline{u})(\overline{U}-\overline{V})+\overline{u}(N-\overline{u})(\overline{U}_+^s-(v-\phi_1)\overline{U}_+^{s-1})$$
$$=\overline{U}_+^s(N-\overline{U}-2\overline{u})(\overline{U}-\overline{V})+\overline{u}(N-\overline{u})(\overline{U}_+^s-\underline{V}\overline{U}_+^{s-1}).$$

As in previous cases we use the equality $\overline{U}_+\overline{U}=\overline{U}_+^2$ to obtain

$$\overline{U}_+^s(N-\overline{U}-2\overline{u})(\overline{U}-\overline{V})+\overline{u}(N-\overline{u})(\overline{U}_+^s-\underline{V}\overline{U}_+^{s-1})$$
$$=\overline{U}_+^s(N-\overline{U}_+-2\overline{u})(\overline{U}_+-\overline{V})+\overline{u}(N-\overline{u})(\overline{U}_+^s-\underline{V}\overline{U}_+^{s-1}).$$

Notice that $(N - \overline{U}_+ - 2\overline{u}) = (N - u - \overline{u}) \le 0$ and the term

$$-\overline{U}_+^s (N - \overline{U}_+ - 2\overline{u})\overline{V}$$

in the above inequality is bounded as in case I, i.e.

$$-\overline{U}_+^s (N - \overline{U}_+ - 2\overline{u})\overline{V} \le -\overline{U}_+^s (N - \overline{U}_+ - 2\overline{u})\overline{V}_+.$$

Thanks to Young's inequality we know that

$$\overline{V}_+\overline{U}_+^s \le k(s)\overline{V}_+^{s+1} + \overline{U}_+^{s+1}$$

and we obtain

$$\begin{aligned}
\overline{U}_+^s (N - \overline{U}_+ - 2\overline{u})(\overline{U}_+ - \overline{V}_+) &= (N - \overline{U}_+ - 2\overline{u})(\overline{U}_+^{s+1} - \overline{V}_+\overline{U}_+^s) \\
&\le (N - \overline{U}_+ - 2\overline{u})(\overline{U}_+^{s+1} - \overline{U}_+^{s+1} - k(s)\overline{V}_+^{s+1}) \le k_2(s)(\overline{V}_+^{s+1} + \overline{U}_+\overline{V}_+^{s+1}) \\
&\le k(\overline{V}_+^{s+1} + \overline{U}_+^s + k_2\overline{V}_+^{s\frac{s+1}{s-1}}).
\end{aligned} \tag{3.19}$$

The remaining term $-\overline{u}(N - \overline{u})\underline{V}\overline{U}_+^{s-1}$ is treated as in case II, i.e.

$$-\overline{u}(N - \overline{u})(\overline{U}_+^s - \underline{V}\overline{U}_+^{s-1}) \le \overline{u}(N - \overline{u})(\overline{U}_+^s + \underline{V}_-\overline{U}_+^{s-1}).$$

Thanks to the assumption $\overline{u} < N$ and Young's inequality we obtain

$$\begin{aligned}
\overline{u}(N - \overline{u})(\overline{U}_+^s + \underline{V}_-\overline{U}_+^{s-1}) &\le \frac{N}{4}(\overline{U}_+^s + \underline{V}_-\overline{U}_+^{s-1}) \\
&\le k(s)(\overline{U}_+^s + \underline{V}_-^s).
\end{aligned} \tag{3.20}$$

Thanks to (3.19) and (3.20), we obtain after integration over Ω

$$\int_\Omega [\chi u(N - u)(u - v) - \chi\overline{u}(N - \overline{u})(\overline{u} - \phi_1(\overline{u}, \underline{u}))]\overline{U}_+^{s-1} \le k \int_\Omega \overline{V}_+^{s+1} + \overline{V}_+^{s\frac{s+1}{s-1}} + \overline{U}_+^s + \overline{V}_-^s. \tag{3.21}$$

Therefore, as a consequence of (3.16), (3.18) and (3.21), the proof of the lemma ends. $\square$

Lemma 3.4. *For any $\epsilon > 0$ arbitrary there exists a positive constant $k(\epsilon)$ such that the following inequality holds:*

$$\begin{aligned}
-\chi \int_\Omega \overline{U}_+^{s-1}(N - 2u)\nabla\overline{U}\nabla v &\le \chi\left(\frac{2}{s+1} + \epsilon\right)\int_\Omega \overline{U}_+^{s+2} + (a_2 + \epsilon)\int_\Omega \overline{U}_+^{s+1} \\
&\quad + k(\epsilon)\int_\Omega (\overline{U}_+^s + \underline{V}_-^{s+2} + \underline{V}_-^{s+1} + \underline{V}_-^s)
\end{aligned} \tag{3.22}$$

for

$$a_2 := \chi \sup_{t>0}\left(\frac{2\overline{u} - N}{s} + \frac{2(\overline{u} - \underline{u})}{s+1}\right).$$

Proof. We multiply the term $-\chi(N - 2u)\nabla\overline{U}\nabla v$ by $\overline{U}_+^{s-1}$ to deduce that

$$\begin{aligned}
-\chi\overline{U}_+^{s-1}(N - 2u)\nabla\overline{U}\nabla v &= -\chi\overline{U}_+^{s-1}(N - 2\overline{U}_+ - 2\overline{u})\nabla\overline{U}_+\nabla v \\
&= -\chi\nabla\left[\left(\frac{N}{s} - \frac{2\overline{u}}{s}\right)\overline{U}_+^s - \frac{2}{s+1}\overline{U}_+^{s+1}\right]\nabla v.
\end{aligned}$$

After space integration, we obtain

$$\begin{aligned}
-\chi\int_\Omega \nabla\left[\left(\frac{N}{s} - \frac{2\overline{u}}{s}\right)\overline{U}_+^s - \frac{2}{s+1}\overline{U}_+^{s+1}\right]\nabla v &= \chi\int_\Omega \left[\left(\frac{N}{s} - \frac{2\overline{u}}{s}\right)\overline{U}_+^s - \frac{2}{s+1}\overline{U}_+^{s+1}\right]\Delta v \\
&= -\chi\int_\Omega \left[\left(\frac{N}{s} - \frac{2\overline{u}}{s}\right)\overline{U}_+^s - \frac{2}{s+1}\overline{U}_+^{s+1}\right](u - v) \\
&= -\chi\int_\Omega \left(\frac{N}{s} - \frac{2\overline{u}}{s} - \frac{2}{s+1}\overline{U}_+\right)\overline{U}_+^s(u - v).
\end{aligned} \tag{3.23}$$

We consider two different cases:

Case 1. $\frac{N}{s} - \frac{2\overline{u}}{s} - \frac{2}{s+1}\overline{U}_+ \geq 0.$
In that case we have

$$-\overline{U}_+^s(u-v) = -\overline{U}_+^s(\overline{U}-\overline{V}) \leq -\left(\overline{U}_+^{s+1} - \overline{U}_+^s\overline{V}_+\right)$$

since

$$-\overline{U}_+^{s+1} + \overline{U}_+^s\overline{V}_+ \leq -\overline{U}_+^{s+1} + \overline{U}_+^{s+1} + k(s)\overline{V}_+^{s+1}$$

and $2\frac{u-\overline{u}}{s+1} < \frac{N}{s}$, we deduce

$$-\chi\left(\frac{N}{s} - \frac{2\overline{u}}{s} - \frac{2}{s+1}(u-\overline{u})\right)\overline{U}_+^s(u-v) \leq k(s)\overline{V}_+^{s+1}. \tag{3.24}$$

Case 2. $\frac{N}{s} - \frac{2\overline{u}}{s} - \frac{2}{s+1}\overline{U}_+ \leq 0.$
In that case we have that

$$-\chi\left(\frac{N}{s} - \frac{2\overline{u}}{s} - \frac{2}{s+1}\overline{U}\right)\overline{U}_+^s(u-v) \leq -\chi\left(\frac{N}{s} - \frac{2\overline{u}}{s} - \frac{2}{s+1}\overline{U}\right)\overline{U}_+^s\left(\overline{U}_+ + \underline{V}_- + \overline{u} - \underline{u}\right)$$
$$\leq a_1\overline{U}_+^{s+2} + a_2\overline{U}_+^{s+1} + a_3\overline{U}_+^s + a_1\underline{V}_-\overline{U}_+^{s+1} + a_2\underline{V}_-\overline{U}_+^s,$$

for

$$a_1 := \frac{2\chi}{s+1}; \qquad a_2 := \chi\sup_{t>0}\left(\frac{2\overline{u}-N}{s} + \frac{2(\overline{u}-\underline{u})}{s+1}\right)$$

and

$$a_3 := \chi\sup_{t>0}\left(\frac{N}{s} - \frac{2\overline{u}}{s}\right)(\overline{u}-\underline{u}).$$

Notice that, thanks to Hölder Inequality we have

$$-\chi\left(\frac{N}{s} - \frac{2\overline{u}}{s} - \frac{2}{s+1}\overline{U}\right)\overline{U}_+^s(u-v) \leq \chi\left(\frac{2}{s+1} + \epsilon\right)\overline{U}_+^{s+2} + (a_2+\epsilon)\overline{U}_+^{s+1}$$
$$+ k(\epsilon)(\overline{U}_+^s + \underline{V}_-^{s+2} + \underline{V}_-^{s+1} + \underline{V}_-^s). \tag{3.25}$$

After integration the proof of the lemma is complete. □

End of the proof of Theorem 3.1
Notice that, by assumption (1.6), we have

$$\int_\Omega (g(u) - g(\overline{u}))\overline{U}_+^{s-1} \leq \lambda\int_\Omega \overline{U}_+^s. \tag{3.26}$$

Then, thanks to (3.12), (3.22) and (3.26), Eq. (3.6) becomes

$$\frac{1}{s}\frac{d}{dt}\int_\Omega \overline{U}_+^s + (s-1)\int_\Omega \overline{U}_+^{s-2}|\nabla\overline{U}_+|^2 \leq \chi\left(\frac{2}{s+1} - 1 + 2\epsilon\right)\int_\Omega \overline{U}_+^{s+2} + (a_2+\epsilon)\int_\Omega \overline{U}_+^{s+1}$$
$$+ k(\epsilon)\int_\Omega \overline{U}_+^s + k(\epsilon)\sum_{j\in J}\int_\Omega \overline{V}_+^j + \underline{V}_-^j \tag{3.27}$$

for $J := \{s\frac{s+1}{s-1}, s+2, s+1, s\}$.
We notice that, for $s > \frac{n}{2}$ and thanks to Lemma 3.2

$$\int_\Omega \overline{V}_+^j \leq C(s,\Omega)\left|\int_\Omega \overline{U}_+^s\right|^{\frac{j}{s}} \quad \text{and} \quad \int_\Omega \overline{V}_-^j \leq C(s,\Omega)\left|\int_\Omega \overline{U}_-^s\right|^{\frac{j}{s}}.$$

We take now

$$\epsilon := \frac{1}{4}$$

and $s \geq 7$, as in (3.5), from (3.27) we have

$$\frac{1}{s}\frac{d}{dt}\int_\Omega \overline{U}_+^s \leq -\frac{\chi}{2}\int_\Omega \overline{U}_+^{s+2} + k_1\int_\Omega \overline{U}_+^{s+1} + k_2\sum_{j\in J}(\|\overline{U}_+\|_{L^s(\Omega)}^j + \|\overline{U}_-\|_{L^s(\Omega)}^j).$$

Since

$$-\frac{\chi}{2}\overline{U}_{+}^{s+2}+k_1\overline{U}_{+}^{s+1}\leq\frac{2k_1^2}{\chi}\overline{U}_{+}^{s}$$

we obtain

$$\frac{1}{s}\frac{d}{dt}\int_{\Omega}\overline{U}_{+}^{s}\leq k_3\sum_{j\in J}(\|\overline{U}_{+}\|_{L^s(\Omega)}^{j}+\|\overline{U}_{-}\|_{L^s(\Omega)}^{j}). \tag{3.28}$$

Notice that $j\geq s$ for any $j\in J$. In the same way we prove

$$\frac{1}{s}\frac{d}{dt}\int_{\Omega}\overline{U}_{-}^{s}\leq k_3\sum_{j\in J}(\|\overline{U}_{+}\|_{L^s(\Omega)}^{j}+\|\overline{U}_{-}\|_{L^s(\Omega)}^{j}). \tag{3.29}$$

We add both expressions to conclude

$$\frac{1}{s}\frac{d}{dt}(\|\overline{U}_{+}\|_{L^s}^{s}+\|\underline{U}_{-}\|_{L^s}^{s})\leq 2k_3\sum_{j\in J}(\|\overline{U}_{+}\|_{L^s}^{j}+\|\underline{U}_{-}\|_{L^s}^{j}).$$

Since $j\geq s$ for any $j\in J$, Gronwall's lemma ends the proof of Theorem 3.1. $\square$

4. Global existence and uniqueness of classical solutions

Taking into account the results obtained in Theorem 3.1, Lemmas 2.2 and 2.3, to have the complete proof of Theorem 1.1 we only need the global existence and uniqueness of solutions of (1.3). Consequently, we establish the existence of smooth solutions to (1.3) as follows:

Theorem 4.1. *We consider $u_0\in C^{2,\alpha}(\overline{\Omega})$, for some $\alpha\in(0,1)$ and we assume that there exists β, such that*

$$0<\beta\leq u_0\quad \text{for } x\in\Omega.$$

Then, for any $T\leq\infty$ there exists a unique classical solution to (1.3)

$$u,v\in C_{x,t}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}_T).$$

Proof. The existence proof follows a standard fixed point argument in $C^0(\overline{\Omega}_T)$ for $T<\infty$. Let S be defined as follows:

$$S:=\left\{u\in C^0(\overline{\Omega}_T)\text{ such that }0\leq u\leq M\right\},$$

where $M:=\max\{1,N,\sup\{u_0\}\}$. Notice that S is a bounded set in $C^0(\overline{\Omega}_T)$. Let $J:S\to C^0(\overline{\Omega}_T)$ defined by

$$J(\tilde{u})=u,$$

where u satisfies the equation

$$u_t-\Delta u=-\chi(\tilde{u}(N-\tilde{u})\nabla\tilde{v})\cdot\nabla u+\chi u(N-u)(u-\tilde{v})+\lambda g(u), \tag{4.1}$$

and $\tilde{v}$ is the solution to the problem

$$-\Delta\tilde{v}+\tilde{v}=\tilde{u},\quad x\in\Omega. \tag{4.2}$$

Notice that $\tilde{v}\in W^{2,p}(\Omega)$ for $p<\infty$, see Agmon, Douglis and Nirenberg [13]. Thanks to Theorem 3.1 and Lemma 2.1, standard theory gives us

$$u\in Y_p:=L^p((0,T);W^{2,p}(\Omega))\cap W^{1,p}((0,T);L^p(\Omega)),\quad \text{for } p<\infty.$$

After routinary computations we see that

$$J:S\to L^p((0,T);W^{2,p}(\Omega))\cap W^{1,p}((0,T);L^p(\Omega))$$

is a continuous function. Since Y_p is compactly embedded in $C^0(\overline{\Omega}_T)$, $J(S)$ is a relatively compact set in $C^0(\overline{\Omega}_T)$ and there exists at least a fixed point of J in S, the solution to the problem. Standard arguments in parabolic equations shows uniqueness of solutions. Since $u\in Y_p$ we have that $u\in C_{x,t}^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}_T)$ and then $v\in C_{x,t}^{2+\alpha,\frac{\alpha}{2}}(\overline{\Omega}_T)$ therefore we have that

$$\nabla v,\quad \chi u(N-u)(u-v)+g(u)\in C_{x,t}^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}_T)$$

since $u_0 \in C^{2,\alpha}(\overline{\Omega})$ we have that, by linear parabolic theory (see for instance [14, Theorem IV. 5.3, p. 320])

$$u \in C_{x,t}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}_T)$$

and also have

$$v \in C_{x,t}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}_T).$$

The proof ends taking limits as $T \to \infty$. $\square$

Corollary 4.2. • *If* $N \leq 1$ *the steady state* $(u, v) \in [L^{\infty}(\Omega)]^2$ *of* (1.3) *satisfying*

$$0 \leq u, \quad 0 \leq v, \tag{4.3}$$

are given by

$$(u, v) = (0, 0) \quad \text{or} \quad (u, v) = (1, 1).$$

• *If* $N > 1$ *and* $\lambda N > 2\chi$, *the unique steady state* $(u, v) \in [L^{\infty}(\Omega)]^2$ *of* (1.3) *satisfying*

$$0 < u, \quad 0 < v, \tag{4.4}$$

is given by

$$(u, v) = (1, 1).$$

Proof. We consider first the case $N \leq 1$, then, after integration over Ω in the first equation of (1.3) we have

$$\int_{\Omega} u = \int_{\Omega} u^2. \tag{4.5}$$

Lemma 2.2 and Theorem 3.1 show that $u \leq 1$ and thanks to (4.3) and (4.5) the proof ends for the case $N \leq 1$.

In a similar way we prove the second case. $\square$

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